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DIRICHLET SERIES IDENTITY INVOLVING THE TOTIENT AND MÖBIUS FUNCTIONS

R. Sivaraman* & J. López-Bonilla**

* Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai, Tamil Nadu, India

** ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

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Abstract:

We use the Ramanujan's sums and an Ivic's formula to show a Dirichlet series identity involving the Möbius and Euler totient functions.

Key Words: Möbius Function, Dirichlet Series, Ramanujan's Sums, Euler Totient Function, Ivic's Identity, Riemann Zeta Function.

1. Introduction:

In [1] is the following identity for a Dirichlet series involving the Möbius [2-6] and totient [2-4, 7, 8] functions:

$$Q(m) = \sum_{n=1}^{\infty} \frac{\mu(\frac{m}{(m,n)})}{\varphi(\frac{m}{(m,n)})} \frac{1}{n^2} = \zeta(2)\frac{(-1)^k}{m^2}\gamma, \qquad \gamma = \prod_{p|m} p\,, \qquad m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\,, \tag{1}$$

Where participate a value of the Riemann zeta function, $\zeta(2) = \frac{\pi^2}{6}$,the core function γ[9]and the number of distinct primes in the prime factorization of m;besides, $(m, n)$is the g.c.d of mand n.

In Sec. 2 we use the Ramanujan's sums [4, 6, 10-13] to show (1).

2. Dirichlet Series Identity:

The Ramanujan's sums are defined by the expression:

$$C(n, m) = \sum_{1 \le k < m,\ (k,m)=1} \exp\left(i\,\frac{2\pi k n}{m}\right), \qquad m \ge 1, \qquad n \ge 0, \tag{2}$$

And Hölder [14] proved the relation:

$$C(n, m) = \frac{\mu(\frac{m}{(m,n)})}{\varphi(\frac{m}{(m,n)})}\varphi(m), \tag{3}$$

Then from (1) and (3):

$$Q(m) = \frac{1}{\varphi(m)} \sum_{n=1}^{\infty} \frac{1}{n^2} C(n, m). \tag{4}$$

On the other hand, we have the following formula obtained by Ivic [15]:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} C(n, m) = \left(\sum_{r=1}^{\infty} \frac{f(r)}{r^s}\right)\left(\sum_{d|m} \mu(\frac{m}{d})\, d^{1-s} f(d)\right), \tag{5}$$

Where fis any totally multiplicative function; thus, in (5) we employ $f = 1$and $s = 2$ to obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} C(n, m) = \zeta(2) \sum_{d|m} \mu(\frac{m}{d})\frac{1}{d} = \frac{\zeta(2)}{m}\sum_{d|m} \mu(d)\, d = \frac{\zeta(2)}{m}\prod_{r=1}^{k}(1 - p_r), \tag{6}$$

$$= (-1)^k \frac{\zeta(2)\,\gamma}{m}\prod_{r=1}^{k}(1 - \frac{1}{p_r}) = \zeta(2)\,\frac{(-1)^k\,\gamma}{m^2}\,\varphi(m),$$

Whose application in (4) gives (1), q.e.d.

Remark 1:

For the special case $m = p_1 p_2 \cdots p_k$, from (1) it is immediate the expression:

$$Q(m) = \zeta(2)\frac{(-1)^k}{m}. \tag{7}$$

Remark 2:

We may indicate an interesting connection between the totient function and the Ramanujan's sums [13]:

$$C(n, m) = \varphi(m) - 2\sum_{r=1,\ (r,m)=1}^{m-1} \sin^2(\frac{\pi r n}{m})\,. \tag{8}$$

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